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A triangle inequality in Hilbert modules over matrix algebras

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Dedicated to Professor T. Ando on the occasion of his 70th birthday

Abstract

The matrix-valued triangle inequalities of R.C. Thompson [Pacific J. Math. 66 (1976) 285–290] are extended to sequences of matrices with real, complex, or quaternion entries. These new matrix inequalities, in turn, imply a natural formulation of the triangle inequality which is valid in certain Hilbert modules over real or complex semisimple matrix algebras. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

In 1976, Thompson [7] proved that if A and B are complex matrices, then there exist unitary matrices U and V such that, with respect to the Löwner partial order, $|A+B| \leqslant U^*|A|U + V^*|B|V$, where $|M|$ denotes the unique positive (semi)definite square root $(M^*M)^{1/2}$ of M^*M, for any complex matrix M. More recently, Lance observed in his monograph [6, p. 4] that if γ and δ are elements of a Hilbert C^*-module, then the triangle inequality $|\gamma+\delta| \leqslant |\gamma| + |\delta|$ need not hold. On the surface, Lance's observation appears to be related to Thompson's matrix-valued in-

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equality, and so it seems reasonable to ask whether a triangle inequality in Hilbert modules might hold if one conjugates by unitaries as Thompson does in $\mathscr{M}_n(\mathbb{C})$. In this paper, we demonstrate that such triangle inequalties do hold in a few special cases of interest. Specifically, we prove that if $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, and if $\mathscr{H}$ is a separable Hilbert space and $\mathscr{A}$ is a finite-dimensional C^*-algebra over $\mathbb{K}$, then for any pair γ and δ in the Hilbert $\mathscr{A}$-module $\mathscr{H} \otimes_{\mathbb{K}} \mathscr{A}$ there are unitaries $u, v \in \mathscr{A}$ such that

$$|\gamma + \delta| \leqslant u^*|\gamma|u + v^*|\delta|v.$$

The gap between Thompson's inequality and our main result above is bridged by formulating a version of Thompson's inequality that is appropriate for sequences of matrices. More precisely, a simple device with matrix algebra allows Thompson's matrix-valued triangle inequality to be extended to several variables.

The first part of this paper formulates the necessary matrix inequalities, and it is only later that the concept of Hilbert module need be introduced. Because applications Hilbert modules frequently occur when the base field is real rather than complex, we develop our results so that they apply to real or complex semisimple matrix algebras. So doing, it is necessary to use matrices with quaternion entries in addition to real and complex matrices.

2. Matrix inequalities

Recall that the skew-field $\mathbb{H}$ of quaternions is a four-dimensional algebra over $\mathbb{R}$ with basis $\{1, i, j, k\}$, where
(a) 1 is the multiplicative identity of $\mathbb{H}$,
(b) $i^2 = j^2 = k^2 = -1$,
(c) $ij = k$, $jk = i$, $ki = j$, and
(d) $ji = -k$, $kj = -i$, $ik = -j$.
Therefore, each $q \in \mathbb{H}$ has a unique representation of the form $q = a_0 1 + a_1 i + a_2 j + a_3 k$ for some $a_0, a_1, a_2, a_3 \in \mathbb{R}$. If $q = a_0 1 + a_1 i + a_2 j + a_3 k$, then the quaternion-conjugate of q is $\overline{q} = a_0 1 - a_1 i - a_2 j - a_3 k$. Observe that

$$\overline{q}q = q\overline{q} = \left(\sum_{j=0}^{3} a_j^2\right) 1,$$

which shows that every nonzero quaternion is invertible in $\mathbb{H}$.

It is natural to regard $\mathbb{R}$ and $\mathbb{C}$ as real subalgebras of $\mathbb{H}$:

$$\mathbb{R} \cong \{a_0 1 : a_0 \in \mathbb{R}\} \quad \text{and} \quad \mathbb{C} \cong \{a_0 1 + a_1 i : a_0, a_1 \in \mathbb{R}\}.$$

However, of greater use is the embedding ϱ of $\mathbb{H}$ as a real subalgebra of $\mathscr{M}_4(\mathbb{R})$ (the algebra of 4×4 real matrices) that arises from the left regular representation of $\mathbb{H}$: namely, $\varrho : \mathbb{H} \to \mathscr{M}_4(\mathbb{R})$, where

$$\varrho(a_0 1 + a_1 i + a_2 j + a_3 k) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}.$$

Denote the transpose operation in $\mathscr{M}_4(\mathbb{R})$ by $*$. Then the (injective) homomorphism ϱ has the property

$$\varrho(\overline{q}) = \varrho(q)^*; \quad q \in \mathbb{H}.$$

Henceforth, $\mathbb{E}$ is to denote either of the fields $\mathbb{R}$ or $\mathbb{C}$, or the skew-field $\mathbb{H}$, and $\mathscr{M}_n(\mathbb{E})$ is the set of $n \times n$ matrices with entries from $\mathbb{E}$. The sets $\mathscr{M}_n(\mathbb{R})$ and $\mathscr{M}_n(\mathbb{H})$ are considered as algebras over $\mathbb{R}$, whereas the set $\mathscr{M}_n(\mathbb{C})$ is viewed as a complex algebra. In all cases, $\mathscr{M}_n(\mathbb{E})$ is an involutive algebra, where the involution $*$ is:
(a) the transpose if $\mathbb{E} = \mathbb{R}$,
(b) the conjugate transpose if $\mathbb{E} = \mathbb{C}$, or
(c) the quaternion-conjugate transpose if $\mathbb{E} = \mathbb{H}$.
The embedding $\varrho : \mathbb{H} \to \mathscr{M}_4(\mathbb{R})$ extends to a faithful representation $\varrho_n : \mathscr{M}_n(\mathbb{H}) \to \mathscr{M}_{4n}(\mathbb{R})$ as follows:

$$\varrho_n\left([q_{st}]_{s,t=1}^n\right) = \left([\varrho(q_{st})]_{s,t=1}^n\right)$$

for all $[q_{st}]_{s,t=1}^n \in \mathscr{M}_n(\mathbb{H})$. Again the important properties of ϱ_n are:
(i) that ϱ_n is injective,
(ii) that ϱ_n is a homomorphism, and
(iii) that $\varrho_n(A^*) = \varrho_n(A)^*$ for all $A \in \mathscr{M}_n(\mathbb{H})$.
Because we are using quaternions, we shall make a minor distinction between the concepts of spectrum and eigenvalue, and we will exercise some care with regards to the partial order on Hermitian matrices by using self-adjointness and spectra rather than actions of matrices on vectors and quadratic forms.

Definition. Let $A, B \in \mathscr{M}_n(\mathbb{E})$, where $\mathbb{E}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.
1. The *spectrum* of A is the set $\sigma(A) \subset \mathbb{C}$ of all roots of the minimal (monic) annihilating polynomial f of A. If $\mathbb{E} = \mathbb{R}$ or $\mathbb{E} = \mathbb{H}$, then $f \in \mathbb{R}[x]$, whereas $f \in \mathbb{C}[x]$ if $\mathbb{E} = \mathbb{C}$.
2. If $A^* = A$, then A is called a *Hermitian* matrix.
3. If $A^* = A$ and $\sigma(A) \subset \mathbb{R}_0^+$ (the nonnegative real numbers), then A is called a *positive* matrix.
4. If A and B are Hermitian, then the notation $A \leqslant B$ means that $B - A$ is positive.
5. If $A^*A \leqslant 1$, then A is called a *contraction*. (Here $1 \in \mathscr{M}_n(\mathbb{E})$ is the identity matrix.)
6. If $A^*A = AA^* = 1$, then A is called a *unitary* matrix.

Observe that in matrix theory it is common to call positive real or complex matrices positive semidefinite. Similarly, the spectrum $\sigma(A)$ of a real or complex matrix A is the set of eigenvalues of A; if $A \in \mathscr{M}_n(\mathbb{H})$, then $\sigma(A)$ is the set of eigenvalues of

$\varrho_n(A)$. With any positive (semidefinite) real or complex $n \times n$ matrix A, we adopt the standard convention of ordering the eigenvalues of A in descending order, $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A) \geqslant 0$, where the number of appearances of a particular eigenvalue λ is equal to the dimension of the kernel $\operatorname{Ker}(A - \lambda 1)$ and is known as the *geometric multiplicity* of λ.

To handle positive matrices with quaternion entries, the following result is useful.

Lemma 2.1. *If $Q \in \mathscr{M}_n(\mathbb{H})$, then Q^*Q is positive. Furthermore, if $A \in \mathscr{M}_n(\mathbb{H})$ is positive, then there are matrices $U, D \in \mathscr{M}_n(\mathbb{H})$ such that*

(i) *U is unitary and D is a diagonal matrix with nonnegative diagonal entries $d_1, d_2, \ldots, d_n$,*
(ii) *$U^*AU = D$,*
(iii) *$\sigma(A) = \{d_1, d_2, \ldots, d_n\}$, and*
(iv) *if $\mu \in \sigma(A)$ appears t_μ times on the diagonal of D, then the geometric multiplicity of μ as an eigenvalue of $\varrho_n(A)$ is $4t_\mu$.*

Proof. Consider the embedding $\varrho_n : \mathscr{M}_n(\mathbb{H}) \to \mathscr{M}_{4n}(\mathbb{R})$. Because ϱ_n is an injective homomorphism, the minimal monic annihilating polynomials of $\varrho_n(A)$ and A agree for every $A \in \mathscr{M}_n(\mathbb{H})$. Hence, $\sigma(A) = \sigma(\varrho_n(A))$ for all $A \in \mathscr{M}_n(\mathbb{H})$. If $Q \in \mathscr{M}_n(\mathbb{H})$, then

$$\varrho_n(Q^*Q) = \varrho_n(Q^*)\varrho_n(Q) = \varrho_n(Q)^*\varrho_n(Q)$$

and, consequently, $\sigma(Q^*Q) = \sigma(\varrho_n(Q)^*\varrho_n(Q)) \subset \mathbb{R}_0^+$. This proves that Q^*Q is positive in $\mathscr{M}_n(\mathbb{H})$.

Suppose that $A \in \mathscr{M}_n(\mathbb{H})$ is positive. Because A is Hermitian, there exist a diagonal matrix $D \in \mathscr{M}_n(\mathbb{R})$, with nonnegative diagonal entries $d_1, d_2, \ldots, d_n$, and a unitary $U \in \mathscr{M}_n(\mathbb{H})$ such that $U^*AU = D$ [3]. Let $\mu_1, \mu_2, \ldots, \mu_m \in \mathbb{R}$ be the distinct real numbers that comprise the list $d_1, d_2, \ldots, d_n$. Because D is a diagonal matrix, the minimal polynomial of D is $f(x) = (x - \mu_1)(x - \mu_2) \ldots (x - \mu_m)$, and therefore $\sigma(A) = \{\mu_1, \mu_2, \ldots, \mu_m\} = \{d_1, d_2, \ldots, d_n\}$.

For each $s = 1, 2, \ldots, n$, set $D_s = \operatorname{diag}\{d_s, d_s, d_s, d_s\} \in \mathscr{M}_4(\mathbb{R})$. Then $U^*AU = D$ implies that

$$\varrho_n(U)^*\varrho_n(A)\varrho_n(U) = \varrho_n(U^*AU) = \varrho_n(D) = D_1 \oplus D_2 \oplus \cdots \oplus D_n. \tag{1}$$

Because $U^*U = 1$ in $\mathscr{M}_n(\mathbb{H})$, $\varrho_n(1) = \varrho_n(U)^*\varrho_n(U)$ in $\mathscr{M}_{4n}(\mathbb{R})$, which means that $\varrho_n(U)$ is unitary. Thus, Eq. (1) is a diagonalisation of the Hermitian matrix $\varrho_n(A)$ by the unitary $\varrho_n(U)$. As a consequence, if $\mu \in \sigma(A)$ appears t_μ times on the diagonal of D, then μ appears $4t_\mu$ times on the diagonal of $D_1 \oplus D_2 \oplus \cdots \oplus D_n$; in other words, the dimension of $\operatorname{Ker}(\varrho_n(A) - \mu 1)$ is $4t_\mu$. $\square$

The following lemma provides the bridge between Thompson's one-variable triangle inequality and the version (Theorem 2.3) of the triangle inequality in several variables.

Lemma 2.2. *Let $\mathbb{E}$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let $A, Z \in \mathscr{M}_n(\mathbb{E})$. If A is positive and if Z is a contraction, then there is a unitary $U \in \mathscr{M}_n(\mathbb{E})$ such that*

$$Z^*AZ \leqslant U^*AU.$$

Moreover, for invertible matrices A, we have that $Z^*AZ = U^*AU$, for some unitary $U \in \mathscr{M}_n(\mathbb{E})$, if and only if Z is unitary and UZ^* commutes with A.

Proof. First assume that $\mathbb{E}$ is the field $\mathbb{C}$. Because Z is a contraction, $1 - Z^*Z \geqslant 0$; hence $1 - Z^*Z$ has a (unique) positive square root $(1 - Z^*Z)^{1/2}$. The matrices Z^*Z and ZZ^* have the same spectrum; thus, $1 - ZZ^*$ is also positive and has, therefore, a positive square root $(1 - ZZ^*)^{1/2}$. Let $V \in \mathscr{M}_{2n}(\mathbb{C})$ be the Halmos unitary [5],

$$V = \begin{bmatrix} Z & (1 - ZZ^*)^{1/2} \\ (1 - Z^*Z)^{1/2} & -Z^* \end{bmatrix}.$$

Then, with $A \oplus 0 \in \mathscr{M}_{2n}(\mathbb{C})$,

$$V^*(A \oplus 0)V = \begin{bmatrix} Z^*AZ & Z^*A(1 - ZZ^*)^{1/2} \\ (1 - Z^*Z)^{1/2}AZ & (1 - ZZ^*)^{1/2}A(1 - ZZ^*)^{1/2} \end{bmatrix}.$$

The Poincaré inequality, [2, pp. 58–59], asserts that each of the first n eigenvalues of $V^*(A \oplus 0)V$ (in decreasing order) dominates the corresponding (ordered) eigenvalue of Z^*AZ; that is,

$$\lambda_s(V^*(A \oplus 0)V) \geqslant \lambda_s(Z^*AZ); \quad s = 1, 2, \ldots, n.$$

But the top n eigenvalues of $V^*(A \oplus 0)V$ are precisely the n eigenvalues of A, including multiplicities. Hence,

$$\lambda_s(A) \geqslant \lambda_s(Z^*AZ); \quad s = 1, 2, \ldots, n. \tag{2}$$

Next we consider the cases where $\mathbb{E}$ is $\mathbb{R}$ or $\mathbb{H}$. The arguments used to obtain inequality (2) are valid for real matrices as well (as every positive $P \in \mathscr{M}_n(\mathbb{R})$ has a unique positive square root $P^{1/2}$, and because Poincaré inequality also holds for real matrices). In the case where A has quaternion entries, use the embedding $\varrho_n : \mathscr{M}_n(\mathbb{H}) \to \mathscr{M}_{4n}(\mathbb{R})$, and then appeal to (2) to obtain

$$\lambda_s(A) \geqslant \lambda_s(\varrho_n(Z)^*\varrho_n(A)\varrho_n(Z)); \quad s = 1, 2, \ldots, 4n. \tag{3}$$

(Note that $\varrho_n(Z)$ is a contraction in $\mathscr{M}_{4n}(\mathbb{R})$.) By using inequality (2) and the Spectral Theorem if $\mathbb{E}$ is $\mathbb{R}$ or $\mathbb{C}$, or by using inequality (3) and Lemma 2.1 if $\mathbb{E} = \mathbb{H}$, we conclude that there exist unitaries $U_1, U_2 \in \mathscr{M}_n(\mathbb{E})$ and diagonal matrices $D_1, D_2 \in \mathscr{M}_n(\mathbb{R})$ such that

(i) $U_1^*(Z^*AZ)U_1 = D_1 = \mathrm{diag}\{d_1^{(1)}, d_2^{(1)}, \ldots, d_n^{(1)}\}$,
(ii) $U_2^*AU_2 = D_2 = \mathrm{diag}\{d_1^{(2)}, d_2^{(2)}, \ldots, d_n^{(2)}\}$, and
(iii) $0 \leqslant d_s^{(1)} \leqslant d_s^{(2)}$ for all $s = 1, 2, \ldots, n$.

Thus, $D_1 \leqslant D_2$ and

$$Z^*AZ = U_1D_1U_1^* \leqslant U_1D_2U_1^* = U_1U_2^*AU_2U_1^* = U^*AU,$$

where $U \in \mathscr{M}_n(\mathbb{E})$ is the unitary matrix $U = U_2^*U_1$.

For the assertion about cases of equality in the inequality, assume that $\mathbb{E} = \mathbb{C}$. Suppose that A is invertible, Z is a contraction, and $Z^*AZ = U^*AU$ for some unitary U. By passing to determinants, we have that

$$\operatorname{Det} A = \operatorname{Det} U^*AU = \operatorname{Det} Z^*AZ = |\operatorname{Det} Z|^2 \operatorname{Det} A.$$

Thus Z has unimodular determinant. Because Z is contractive, the characteristic values of Z are not larger than 1 in modulus. On the other hand, the determinant of Z has modulus 1. Thus, every eigenvalue of Z has modulus 1, which implies that Z is reduced by each of its eigenvectors. Hence, Z is unitary and, consequently, UZ^* commutes with A.

The cases of equality for matrices with real and quaternion entries are handled in a similar fashion and, therefore, the proofs are omitted. $\square$

Definition (*Convergent Series*). If ξ is a vector in $\mathbb{C}^n$ or $\mathbb{R}^n$, then $\|\xi\|$ shall denote its Euclidean norm.

(a) If $\{A_s\}_{s\in\mathbb{N}}$ is a sequence of real or complex $n \times n$ matrices, then $\sum_{s\in\mathbb{N}} A_s^* A_s$ is convergent if $\sum_{s\in\mathbb{N}} \|A_s\xi\|^2$ converges for all $\xi \in \mathbb{C}^n$ (or $\mathbb{R}^n$ if the matrices A_s are real).

(b) If $\{A_s\}_{s\in\mathbb{N}}$ is a sequence in $\mathscr{M}_n(\mathbb{H})$, then $\sum_{s\in\mathbb{N}} A_s^* A_s$ is convergent if $\sum_{s\in\mathbb{N}} \|\varrho_n(A_s)\xi\|^2$ converges for all $\xi \in \mathbb{R}^{4n}$.

Note that if $\mathbb{E}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and if $\{A_s\}_{s\in\mathbb{N}}$ is a sequence of matrices such that $\sum_{s\in\mathbb{N}} A_s^* A_s$ is convergent in $\mathscr{M}_n(\mathbb{E})$, then the matrix $\sum_{s\in\mathbb{N}} A_s^* A_s$ is positive.

The following theorem is the principal result of the present paper.

Theorem 2.3. *Let $\{A_s\}_{s\in\mathbb{N}}$ and $\{B_s\}_{s\in\mathbb{N}}$ be two sequences in $\mathscr{M}_n(\mathbb{E})$, where $\mathbb{E}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. If $\sum_{s\in\mathbb{N}} A_s^* A_s$ and $\sum_{s\in\mathbb{N}} B_s^* B_s$ are convergent, then there exist unitary matrices $U, V \in \mathscr{M}_n(\mathbb{E})$ such that*

$$\left(\sum_{s\in\mathbb{N}} (A_s + B_s)^*(A_s + B_s)\right)^{1/2} \leqslant U^*\left(\sum_{s\in\mathbb{N}} A_s^* A_s\right)^{1/2} U + V^*\left(\sum_{s\in\mathbb{N}} B_s^* B_s\right)^{1/2} V.$$

Proof. Assume that $\mathbb{E} = \mathbb{C}$ and consider the Hilbert space $\mathscr{H} = \mathbb{C}^n \otimes l^2(\mathbb{N})$. Let A and B be the following operators on $\mathscr{H}$:

$$A = \begin{bmatrix} A_1 & 0 & 0 & \dots \\ A_2 & 0 & 0 & \dots \\ A_3 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & 0 & 0 & \dots \\ B_2 & 0 & 0 & \dots \\ B_3 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then

$$[(A+B)^*(A+B)]^{1/2} = \begin{bmatrix} \left[\sum_{s\in\mathbb{N}}(A_s+B_s)^*(A_s+B_s)\right]^{1/2} & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$A^*A = \begin{bmatrix} (\sum_{s\in\mathbb{N}} A_s^*A_s)^{1/2} & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

$$B^*B = \begin{bmatrix} (\sum_{s\in\mathbb{N}} B_s^*B_s)^{1/2} & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

By Thompson's triangle inequality for Hilbert space operators [1, Theorem 2.2], there are isometries R and S acting on $\mathscr{H}$ such that, in $\mathscr{B}(\mathscr{H})$,

$$[(A+B)^*(A+B)]^{1/2} \leqslant R(A^*A)^{1/2}R^* + S(B^*B)^{1/2}S^*.$$

The equality above holds in $\mathscr{M}_n(\mathbb{C})$ for the (1, 1)-entries of these operators, namely,

$$\left(\sum_{s\in\mathbb{N}}(A_s+B_s)^*(A_s+B_s)\right)^{1/2} \leqslant R_{11}\left(\sum_{s\in\mathbb{N}} A_s^*A_s\right)^{1/2} R_{11}^* + S_{11}\left(\sum_{s\in\mathbb{N}} B_s^*B_s\right)^{1/2} S_{11}^*.$$

The matrices R_{11} and S_{11} are contractions (because R and S are contractions). By Lemma 2.2, there are unitary matrices $U, V \in \mathscr{M}_n(\mathbb{C})$ such that

$$R_{11}\left(\sum_{s\in\mathbb{N}} A_s^*A_s\right)^{1/2} R_{11}^* \leqslant U^*\left(\sum_{s\in\mathbb{N}} A_s^*A_s\right)^{1/2} U$$

and

$$S_{11}\left(\sum_{s\in\mathbb{N}} B_s^*B_s\right)^{1/2} S_{11}^* \leqslant V^*\left(\sum_{s\in\mathbb{N}} B_s^*B_s\right)^{1/2} V.$$

Consequently,

$$\left(\sum_{s\in\mathbb{N}}(A_s+B_s)^*(A_s+B_s)\right)^{1/2} \leqslant U^*\left(\sum_{s\in\mathbb{N}} A_s^*A_s\right)^{1/2} U + V^*\left(\sum_{s\in\mathbb{N}} B_s^*B_s\right)^{1/2} V.$$

To obtain the inequality above for real or quaternion matrices, one has simply to note that the proof of Theorem 2.2 in [1] carries over to operators A and B acting on real Hilbert spaces. Therefore, for real matrices, the arguments above apply directly; for quaternion matrices, the arguments above apply after $\mathscr{M}_n(\mathbb{H})$ has been embedded into $\mathscr{M}_{4n}(\mathbb{R})$, via ϱ_n. $\square$

3. Module inequalities

The purpose of this section is to place Theorem 2.3 in a context where the parallels with Thompson's original theorems (see [7,8]) are clearly drawn.

All algebras are henceforth assumed to have a multiplicative identity 1, and $\mathbb{K}$ denotes either of the fields $\mathbb{R}$ or $\mathbb{C}$. An algebra $\mathscr{A}$ over $\mathbb{K}$ with involution $*$ is called a *C^*-algebra* if $\mathscr{A}$ is a Banach algebra and if, for every $a \in \mathscr{A}$, $\|a^*a\| = \|a\|^2$ and $(1 + a^*a)^{-1}$ exists. In a C^*-algebra $\mathscr{A}$, an element $a \in \mathscr{A}$ is said to be *positive* if $a = b^*b$ for some $b \in \mathscr{A}$. Each positive element of $\mathscr{A}$ has a unique positive square root; thus, we write $|c|$ to denote $(c^*c)^{1/2}$, the positive square root of $c \in \mathscr{A}$.

If $\mathscr{A}$ is a C^*-algebra over $\mathbb{K}$, then an *inner-product $\mathscr{A}$-module* is a vector space $\mathscr{E}$ over $\mathbb{K}$ such that $\mathscr{E}$ is a right module (i.e., there is a multiplication $\mathscr{E} \times \mathscr{A} \to \mathscr{E}$ such that $\lambda(\xi a) = (\lambda\xi)a = \xi(\lambda a)$ for all $\xi \in \mathscr{E}$, $a \in \mathscr{A}$ and $\lambda \in \mathbb{K}$) coupled with a function $\langle\cdot,\cdot\rangle : \mathscr{E} \times \mathscr{E} \to \mathscr{A}$ whereby for all $\xi, \eta \in \mathscr{E}$ and $a \in \mathscr{A}$:

(i) $\langle\cdot,\cdot\rangle$ is $\mathbb{K}$-linear in the second variable,
(ii) $\langle\xi, \eta a\rangle = \langle\xi, \eta\rangle a$,
(iii) $\langle\xi, \eta\rangle = \langle\eta, \xi\rangle^*$, and
(iv) $\langle\xi, \xi\rangle$ is positive and $\langle\xi, \xi\rangle = 0$ if and only if $\xi = 0$.

The function $\langle\cdot,\cdot\rangle$ behaves like an inner product on $\mathscr{E}$ except that here the values of the inner product are in $\mathscr{A}$ rather than in $\mathbb{K}$. For elements $\xi \in \mathscr{E}$, define $|\xi|$ by

$$|\xi| = \langle\xi, \xi\rangle^{1/2}. \tag{4}$$

Furthermore, the formula $\|\xi\| = \|\langle\xi, \xi\rangle\|^{1/2}$ defines a norm on $\mathscr{E}$, under which the Cauchy–Schwarz inequality $\|\langle\xi, \eta\rangle\| \leqslant \|\xi\|\|\eta\|$ holds [6, Chapter 1].

An inner-product $\mathscr{A}$-module $\mathscr{E}$ is called a *Hilbert C^*-module* over $\mathscr{A}$ if $\mathscr{E}$ is a Banach space in the norm described above. Every inner-product $\mathscr{A}$-module can be completed so as to become as Hilbert $\mathscr{A}$-module (see [6, p. 4]).

Modules of the form $\mathscr{H} \otimes_{\mathbb{K}} \mathscr{A}$, where $\mathscr{H}$ is a Hilbert space over $\mathbb{K}$, have a particularly important role in the theory of Hilbert C^*-modules. First, consider the algebraic tensor product $\mathscr{H} \otimes_{\mathrm{alg}} \mathscr{A}$, which is a vector space over $\mathbb{K}$. Fix $b \in \mathscr{A}$ and consider the map $r_b : \mathscr{H} \times \mathscr{A} \to \mathscr{H} \otimes_{\mathrm{alg}} \mathscr{A}$ whereby $r_b(\xi, a) = \xi \otimes ab$ for all $(\xi, a) \in \mathscr{H} \times \mathscr{A}$. Then r_b is bilinear and so, by the universal properties of tensor products [4, Chapter 6], there is a unique linear map R_b on $\mathscr{H} \otimes_{\mathrm{alg}} \mathscr{A}$ such that $R_b(\xi \otimes a) = \xi \otimes ab$ for all elementary tensors $\xi \otimes a \in \mathscr{H} \otimes_{\mathrm{alg}} \mathscr{A}$. Hence, $\mathscr{A}$ acts on $\mathscr{H} \otimes_{\mathrm{alg}} \mathscr{A}$ from the right and, for every $b \in \mathscr{A}$,

$$(\xi \otimes a)b = \xi \otimes ab; \quad \xi \in \mathscr{H}, \ a \in \mathscr{A}. \tag{5}$$

Similarly, one obtains an $\mathscr{A}$-valued inner product $\langle \cdot, \cdot \rangle$ on $\mathscr{H} \otimes_{\mathrm{alg}} \mathscr{A}$ under which

$$\langle a \otimes \xi, b \otimes \eta \rangle = \langle \xi, \eta \rangle_{\mathscr{H}} a^* b. \tag{6}$$

Thus, $\mathscr{H} \otimes_{\mathrm{alg}} \mathscr{A}$ is an inner-product $\mathscr{A}$-module; let $\mathscr{H} \otimes_{\mathbb{K}} \mathscr{A}$ denote its completion in the norm $\|\gamma\| = \|\langle \gamma, \gamma \rangle\|^{1/2}$ for $\gamma \in \mathscr{H} \otimes_{\mathrm{alg}} \mathscr{A}$.

Thompson's matrix-valued triangle inequalities take the form below for the following Hilbert modules.

Theorem 3.1. *Let $\mathbb{K}$ be either of the fields $\mathbb{R}$ or $\mathbb{C}$, and suppose that, over $\mathbb{K}$, $\mathscr{A}$ is a finite-dimensional C^*-algebra and $\mathscr{H}$ is a separable Hilbert space. Then for every γ, δ in the Hilbert $\mathscr{A}$-module $\mathscr{H} \otimes_{\mathbb{K}} \mathscr{A}$, there are unitaries $u, v \in \mathscr{A}$ such that*

$$|\gamma + \delta| \leqslant u^*|\gamma|u + v^*|\delta|v.$$

Proof. Let Λ be an index set of cardinality equal to the dimension of $\mathscr{H}$. If $\{\phi_s\}_{s \in \Lambda}$ is an orthonormal basis of $\mathscr{H}$, then for any $\gamma, \delta \in \mathscr{H} \otimes_{\mathbb{K}} \mathscr{A}$ there are sequences $\{g_s\}_{s \in \Lambda}$, $\{d_s\}_{s \in \Lambda}$ in $\mathscr{A}$ such that

$$\gamma = \sum_{s \in \Lambda} \phi_s \otimes g_s \quad \text{and} \quad \delta = \sum_{s \in \Lambda} \phi_s \otimes d_s.$$

Hence,

$$|\gamma + \delta| = \left(\sum_{s \in \Lambda} (g_s + d_s)^* (g_s + d_s) \right)^{1/2} \tag{7}$$

and

$$|\gamma| = \left(\sum_{s \in \Lambda} g_s^* g_s \right)^{1/2}, \quad |\delta| = \left(\sum_{s \in \Lambda} d_s^* d_s \right)^{1/2}. \tag{8}$$

Now if $\mathscr{A}$ is any of the C^*-algebras $\mathscr{M}_n(\mathbb{R})$, $\mathscr{M}_n(\mathbb{C})$, $\mathscr{M}_n(\mathbb{H})$, then, by Theorem 2.3, there exist unitaries $u, v \in \mathscr{A}$ such that

$$|\gamma + \delta| \leqslant u^*|\gamma|u + v^*|\delta|v.$$

In general, however, $\mathscr{A}$ will be an algebraic direct sum $\mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \cdots \oplus \mathscr{A}_m$ of simple C^*-algebras. But this effects a direct sum decomposition of $\mathscr{H} \otimes_{\mathbb{K}} \mathscr{A}$: $(\mathscr{H} \otimes_{\mathbb{K}} \mathscr{A}_1) \oplus \cdots \oplus (\mathscr{H} \otimes_{\mathbb{K}} \mathscr{A}_m)$. As each $\mathscr{A}_s$ $(s = 1, 2, \ldots, m)$ is of the form $\mathscr{M}_{n_s}(\mathbb{C})$ in the case of $\mathbb{K} = \mathbb{C}$, or of the form $\mathscr{M}_{n_s}(\mathbb{R})$, $\mathscr{M}_{n_s}(\mathbb{C})$, or $\mathscr{M}_{n_s}(\mathbb{H})$ in the case $\mathbb{K} = \mathbb{R}$ [4, Chapter 5], the triangle inequality in matrix algebras extends to finite direct sums of matrix algebras. Hence, for every $\gamma, \delta \in (\mathscr{H} \otimes_{\mathbb{K}} \mathscr{A}_1) \oplus \cdots \oplus (\mathscr{H} \otimes_{\mathbb{K}} \mathscr{A}_m)$, there are unitaries $u, v \in \mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \cdots \oplus \mathscr{A}_m$ such that $|\gamma + \delta| \leqslant u^*|\gamma|u + v^*|\delta|v$. $\square$

A second example of a module-type triangle inequality occurs in a purely algebraic setting. For brevity, we shall consider only the case of complex algebras.

A complex algebra $\mathscr{A}$—not necessarily a normed algebra—is *locally finite* if for every finite subset $\mathscr{S} \subset \mathscr{A}$, the smallest subalgebra $\mathscr{B} \subset \mathscr{A}$ that contains $\mathscr{S}$ is finite-dimensional. If a complex algebra $\mathscr{A}$ has an involution $*$, then the involution is said to be *positive* when $a^*a = 0$ if and only if $a = 0$. (Notice that the involution of a C^*-algebra is one example of a positive involution.) The definitions of spectrum, positive, square root, and unitary apply in locally finite algebras with positive involution.

Likewise, the concept of inner-product module extends to locally finte complex algebras $\mathscr{A}$ with positive involution: using (4)–(6), with $\mathscr{H} = \mathbb{C}^m$, the vector space $\mathbb{C}^m \otimes \mathscr{A}$ is an inner-product $\mathscr{A}$-module. (The only essential difference between this concept and the C^*-module concept is that in the present situation the algebra $\mathscr{A}$ need not be normed and the Hilbert space H has finite dimension.)

Theorem 3.2. *Suppose that $\mathscr{A}$ is a locally finite complex algebra with positive involution $*$. Then, for every $\gamma,\ \delta$ in the inner-product $\mathscr{A}$-module $\mathbb{C}^m \otimes \mathscr{A}$, there are unitaries $u, v \in \mathscr{A}$ such that*

$$|\gamma + \delta| \leqslant u^*|\gamma|u + v^*|\delta|v.$$

Proof. If $\gamma, \delta \in \mathbb{C}^m \otimes \mathscr{A}$, then there are $g_1, g_2, \ldots, g_m, d_1, d_2, \ldots, d_m \in \mathscr{A}$ such that $|\gamma + \delta|$, $|\gamma|$, and $|\delta|$ are determined by $g_s,\ d_t$ $(1 \leqslant s, t \leqslant m)$, as in (7) and (8). Let $\mathscr{B}$ be the $*$-algebra generated by $\{g_s, d_t : 1 \leqslant s, t \leqslant m\}$. By hypothesis, $\mathscr{B}$ is finite-dimensional and the induced involution $*$ on $\mathscr{B}$ is positive. Therefore [4, Theorem 5.13] shows that $\mathscr{B}$ can be realised as a C^*-subalgebra of $\mathscr{M}_n(\mathbb{C})$ for some $n \in \mathbb{Z}^+$. Hence, $\mathscr{B}$ contains the (unique) positive square root of any of its elements. Thus, by Theorem 3.1, there are unitaries $u, v \in \mathscr{B}$ such that

$$\begin{aligned}&\left(\sum_{s=1}^{m}(g_s + d_s)^*(g_s + d_s)\right)^{1/2}\\ &\quad\leqslant u^*\left(\sum_{s=1}^{m} g_s^* g_s\right)^{1/2} u + v^*\left(\sum_{s=1}^{m} d_s^* d_s\right)^{1/2} v.\end{aligned}$$

That is, $|\gamma + \delta| \leqslant u^*|\gamma|u + v^*|\delta|v$ for some unitaries $u, v \in \mathscr{B} \subset \mathscr{A}$. $\square$

Surely further triangle inequalities should be possible, particulary for Hilbert $\mathscr{A}$-modules of the form $\mathscr{H} \otimes_{\mathbb{C}} \mathscr{A}$. However, in what might seem to be the easiest case—namely that in which $\mathscr{H} = \mathbb{C}$—it is an open problem as to whether Thompson's triangle inequality holds for every C^*-algebra $\mathscr{A}$ (see [9, p. 34]). More promising, perhaps, is the search for a version of the triangle inequality (in the form of Theorem 3.1) that holds in $\mathscr{H} \otimes_{\mathbb{C}} \mathscr{A}$, where $\mathscr{A}$ is a finite von Neumann algebra. In this case, all that one really requires is a suitable generalisation of Lemma 2.2 to $\mathscr{A}$.

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